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ON EQUILIBRIUM PATTERNS OF FLUID BODIES SITUATED AT LIBRATION POINTS*

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The motion of a fluid body is considered in a field of Newtonian attraction of two bodies circulating about a common center of mass along Keplerian circular orbits. Particular solutions of the problem are obtained on the assumption that the dimensions of the fluid body are small in comparison with the distances between the bodies. To these solutions correspond positions of the body center of mass at libration points of the limited circular problem of three bodies, a fixed position of the principal central axes of inertia in an orbital coordinate system, and ellipsoidal equilibrium patterns.

The first steps towards formulation of the general problem of motion of a system of deformable bodies gravitating to each other were taken in /1/. The basic idea is that translational motions of celestial bodies, the rotational motions of their axes of inertia, and motions of individual particles of these bodies are interrelated and must be generally investigated jointly. The first integrals of the problem of deformable bodies (integrals of momentum and of moment of momentum which generalize classic integrals) were obtained in /1/, where cases of existence of the energy integral were also indicated.

1. The equations of motion of a fluid body. Consider the motion of a homogeneous fluid body M in the field of Newtonian attraction of two bodies M_1 and M_2 moving around their common center of mass G on Keplerian circular orbits. We shall consider bodies M_1, M_2 as material points, and assume that the mass of body M is negligibly small in comparison with masses m_1 and m_2 of basic bodies M_1 and M_2. Particles of the fluid body interact with each other in conformity with Newton's law.

Let $GXYZ$ be an inertial coordinate system with origin at the center of mass of bodies M_1, M_2 and axes GX, GY in the plane of the basic bodies orbit; let $Gxyz$ be a rotating coordinate system whose axis Gz coincide with axis GZ and axis Gx coincide with line M_1M_2 and be directed toward body M_2; $OXYZ$ be a coordinate system with origin at the center of mass O of the fluid body and axes parallel to the respective coordinate axes $GXYZ$, and $O\xi\eta\zeta$ be the "proper" coordinate system of body M whose axes are oriented along its instantaneous principal central axes of inertia.

We define the motion of the body by the following coordinates: x_0, y_0, z_0 the coordinates of its center of mass O in the fixed axes, Euler's angles ψ, θ, φ which determine the respective orientation of coordinate axes $O\xi\eta\zeta$ and $OXYZ$, coordinates and projections $\xi, \eta, \zeta, u, v, w$ of the relative velocity of individual particle of the body in the system of coordinates $O\xi\eta\zeta$, and projections p, q and r of the angular velocity of rotation of the proper coordinate system on axes $O\xi, O\eta, O\zeta$.

The equations of motion of body M are obtained using the principle of least action, as formulated by Hamilton–Ostrogradskii /2/, and expressed in variables $x_0, y_0, z_0, p, q, r, u, v, w, \xi, \eta, \zeta$ are of the form

$$x_0^{\cdot\cdot} = \frac{1}{m}\frac{\partial U}{\partial x_0} \tag{1.1}$$

$$\frac{d}{dt}(Ap+P) + qR - rQ + (C-B)qr = \left(\frac{\partial U}{\partial \psi} - \cos\theta\frac{\partial U}{\partial \varphi}\right)\frac{\sin\varphi}{\sin\theta} + \cos\varphi\frac{\partial U}{\partial \theta} \tag{1.2}$$

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \xi} + v\frac{\partial u}{\partial \eta} + w\frac{\partial u}{\partial \zeta} + 2(qw - rv) = \frac{\partial U'}{\partial \xi} - \frac{1}{\rho_0}\frac{\partial D}{\partial \xi} + \\ \eta r^{\cdot} - \zeta q^{\cdot} - (x_0^{\cdot\cdot}a_{11} + y_0^{\cdot\cdot}a_{21} + z_0^{\cdot\cdot}a_{31}) - (x_0^{\cdot}a_{13} + y_0^{\cdot}a_{23} + z_0^{\cdot}a_{33})q + \\ (x_0^{\cdot}a_{12} + y_0^{\cdot}a_{22} + z_0^{\cdot}a_{32})r - p(\xi p + \eta q + \zeta r) + \xi\Omega^2\end{aligned} \tag{1.3}$$

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$$m=\rho_0\int d\tau,\quad \rho_0=\text{const},\quad \Omega=\sqrt{p^2+q^2+r^2},\qquad A=\rho_0\int(\eta^2+\zeta^2)\,d\tau,\quad B=\rho_0\int(\xi^2+\zeta^2)\,d\tau$$

$$C=\rho_0\int(\eta^2+\xi^2)\,d\tau,\quad P=\rho_0\int(\eta w-\zeta v)\,d\tau,\qquad Q=\rho_0\int(\zeta u-\xi w)\,d\tau,\quad R=\rho_0\int(\xi v-\eta u)\,d\tau$$

The equations for y_0, z_0, q, r, v, w are obtained from Eqs.(1.1)—(1.3) by cyclic permutation of respective variables. The system of equations is closed by the addition of the continuity equation $\partial u/\partial\xi+\partial v/\partial\eta+\partial w/\partial\zeta=0$ and of Euler's kinematic equations /3/.

In Eqs.(1.1)—(1.3) Ω is the angular velocity of rotation of the body axes of inertia, A, B, C are the principal central moments of inertia that correspond to axes $O\xi, O\eta, O\zeta$ of the body, P, Q, R are projections of the moment of momentum of fluid particles in the "proper" coordinate system on axes $O\xi, O\eta, O\zeta$, a_{ij} $(i, j=1, 2, 3)$ are the directional cosines of axes $O\xi\eta\zeta$ in the system of coordinates $OXYZ$ which are known functions of Euler's angles /3/, m is the mass of the fluid body, and D is the pressure. Integration is extended over the whole volume τ occupied by body M at a given instant of time.

The force function U of Newtonian interaction of the fluid body with the basic bodies M_1 and M_2 is defined by the expansion

$$U=f\sum_{i=1}^{2}\left\{\frac{m_i m}{r_i}+\frac{m_i}{2r_i^3}[B+C-2A+3(B-A)\beta_i^2+3(C-A)\gamma_i^2]\right\}+W \tag{1.4}$$

$$r_i^2=(x_0-x_i)^2+(y_0-y_i)^2+z_0^2$$
$$\beta_i=a_{11}\frac{x_0-x_i}{r_i}+a_{21}\frac{y_0-y_i}{r_i}+a_{31}\frac{z_0}{r_i}$$
$$\gamma_i=a_{13}\frac{x_0-x_i}{r_i}+a_{23}\frac{y_0-y_i}{r_i}+a_{33}\frac{z_0}{r_i}$$

where f is the gravitational constant, r_i is the distance between the center of mass of body M and body M_i $(i=1, 2)$, β_i, γ_i are cosines of angles between the straight line M_iO and the axes of inertia $O\eta, O\zeta$, and x_i, y_i $(z_i=0)$ are coordinates of body M_i in the inertial coordinate system. Function W combines all higher harmonics, beginning with the third, of expansion U.

Function U' defines the Newtonian interaction of an arbitrary particle of body M that occupies volume $d\tau$ of the body with its remaining components and material points M_1, M_2.

$$U'=\sum_{i=1}^{2}U_i'+U_0',\quad U_i'=\frac{fm_i}{R_i}$$
$$R_i^2=(x-x_i)^2+(y-y_i)^2+z^2,\quad U_0'=f\rho_0\int\frac{d\tau}{R_0}$$

where R_i is the distance of the considered point of the fluid body with coordinates x, y, z from body M_i in the inertial coordinate system, and R_0 is the distance between two arbitrary points of body M at coordinates x', y', z' and x, y, z in axes $GXYZ$, respectively, i.e.

$$x=x_0+a_{11}\xi+a_{12}\eta+a_{13}\zeta\ldots$$
$$x'=x_0+a_{11}\xi'+a_{12}\eta'+a_{13}\zeta'\ldots$$

where ξ, η, ζ and ξ', η', ζ' are coordinates of the indicated points in "proper" axes. Integration in U_0' is extended over the whole volume occupied by body M.

Let us point out some of the singularities of Eqs.(1.1)—(1.3), which constitute a system of integro-differential equations. The motion of individual particles of the body is related to its principal central axes of inertia. The three basic kinds of motions of the fluid body, viz. translational, rotating and deformation are interdependent /1/. Equations (1.1)—(1.3) admit one first integral which is a generalization of the classic Jacobi's integral for the limited circular problem of three material points or that of two points and an absolutely rigid body.

2. Equilibrium patterns of a fluid body situated at rectilinear libration points. We shall aim at finding such particular solutions of Eqs.(1.1)—(1.3) according to which body M moves as an absolutely rigid body. In such motions the particles of the fluid body are stationary relative to its principal central axes of inertia, i.e. $u=v=w=0$, $P=Q=R=0$ and the moments of interia A, B, C are constant.

Let us make the simplifying assumption that the dimensions of the body are small in comparison with the distances OM_1 and OM_2. This enables us to introduce in the investigation the small parameter $\mu = \bar{R}/\bar{r}$, where $\bar{R}$ is the largest dimension of the fluid body and $\bar{r}$ the minimum distance to the attracting bodies M_1, M_2.

We disregard in the right-hand sides of equations of motion the terms of order μ. The equations of translational motion are then separated from Eqs.(1.2) and (1.3), assuming the form of equations of the limited circular problem of three bodies (three material points). Equations (1.2) and (1.3) remain interconnected and become integrable only after some solution of Eqs.(1.1) has been obtained for $\mu = 0$. Note that when $\mu = 0$ the terms calculated for the approximate value of the force function U, determined by formula (1.4) with $W = 0$, are retained in the right-hand sides of Eqs.(1.2).

Taking into account the assumptions made above, Eqs.(1.1) and (1.2) admit the stationary solutions

$$x_0 = \alpha_i \cos nt, \quad y_0 = \alpha_i \sin nt, \; z_0 = 0, \quad n^2 = f(m_1 + m_2)a_0^{-3/2} \qquad (2.1)$$
$$p = 0, \quad q = n, \quad r = 0, \quad \theta = \pi/2, \quad \varphi = 0, \quad \psi = nt$$

where a_0 is the radius of the orbit of basic bodies, α_i is the constant coordinate of mass center of body M on the rotating axis Gx that corresponds to one of the three libration points ($\alpha_1 = -a_0\rho_1$, $\alpha_2 = a_0\rho_2$, $\alpha_3 = a_0(1 + \rho_3)$), where ρ_1, ρ_2, ρ_3 are constant quantities which are determined by solving known algebraic equations, depending on the single parameter $\nu = m_2/m_1$ /3/).

Solution (2.1) shows that the center of mass of the fluid body is located at one of the rectilinear libration points, and that its principal central axes of inertia have a fixed position in the rotating axes $Gxyz$. The body then rotates at constant angular velocity n (equal to the mean /velocity/ of the orbital motion of the basic bodies) relative to axis $O\eta$ of the orthogonal orbit plane.

It remains to determine whether any solutions of Eq.(1.3) are also valid for variables (2.1) when $\mu = 0$. We shall show that on certain assumptions about the fluid body form, such solutions exist.

Let us assume that body M is a homogeneous ellipsoid with semiaxes a, b, c, (setting for definiteness $a > c > b$), corresponding to inertia axes $O\xi, O\eta, O\zeta$, and find out what constraints are to be imposed on the problem parameters, if Eqs.(1.3) are to admit ellipsoidal equilibrium forms for solution (2.1).

For this we write the expression for the force function U_0' of the homogeneous ellipsoid for the internal point (ξ, η, ζ) /3/

$$U_0' = f(F_0 - F_1\xi^2 - F_2\eta^2 - F_3\zeta^2) \qquad (2.2)$$

$$F_0 = \beta\int_0^\infty \frac{ds}{R(s)}, \quad F_\lambda = \beta\int_0^\infty \frac{ds}{(\lambda^2 + s)R(s)}, \quad \lambda = a, b, c \qquad (2.3)$$
$$\beta = \pi abc\rho_0, \quad R(s) = [(a^2 + s)(b^2 + s)(c^2 + s)]^{1/2}$$

When $\mu = 0$, the force functions U_1', U_2' are determined by formulas

$$U_i' = \frac{1}{2r_i^3}[2r_i^2 - 2\xi r_i - r^2 + 3\zeta^2] \quad (i = 1, 2), \quad r = (\xi^2 + \eta^2 + \zeta^2)^{1/2}$$

where r_i is the distance of body M_i from the corresponding libration point and r is the distance of the particle of fluid at coordinates ξ, η, ζ from the center of mass of body M.

It is then possible to show that in the case of solution (2.1) and $\mu = 0$ the hydrodynamic equations (1.3) admit the first integral

$$D/\rho_0 = \Phi_1\xi^2 + \Phi_2\eta^2 + \Phi_3\zeta^2 + \Phi_4\xi + \text{const} \qquad (2.4)$$

$$\Phi_1 = \frac{1}{2}n^2 + f\left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right) - fF_1 \qquad (2.5)$$

$$\Phi_2 = -\frac{1}{2}f\left(\frac{m_1}{r_1^2} + \frac{m_2}{r_2^3}\right) - fF_2$$
$$\Phi_3 = \frac{1}{2}n^2 - \frac{1}{2}f\left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right) - fF_3$$
$$\Phi_4 = n^2\alpha_i - f\left(\frac{m_1}{r_1^2} + \frac{m_2}{r_2^2}\right)$$

For the existence of ellipsoidal equilibrium patterns it is necessary that the isobar

D = const defined by Eqs.(2.4) and (2.5) coincides with the external surface of the fluid ellipsoid surface. For this it is sufficient that the equalities

$$\Phi_1 a^2 = \Phi_2 b^2 = \Phi_3 c^2 \tag{2.6}$$

$$\Phi_4 = 0 \tag{2.7}$$

be satisfied.

Condition (2.7) is satisfied by virtue of solution (2.1) for the orbital motion of the body /3/.

Conditions (2.6) together with formulas (2.5) define new sequencies of ellipsoidal equilibrium patterns for each of the rectilinear libration points, with the obtained equilibrium patterns becoming Roche's ellipsoids when $\nu = 0$ $(m_2 = 0)$.

3. Equilibrium patterns of a fluid body at triangular libration points.

When $u = v = w = 0$, $\mu = 0$, Eqs.(1.1) and (1.2) admit one more stationary solution

$$x_0 = \frac{a_0(1-\nu)}{2(1+\nu)}\cos nt - \frac{a_0\sqrt{3}}{2}\sin nt, \quad y_0 = \frac{a_0(1-\nu)}{2(1+\nu)}\sin nt + \frac{a_0\sqrt{3}}{2}\cos nt, \quad z_0 = 0 \tag{3.1}$$

$$p = 0, \quad q = n, \quad r = 0, \quad \theta = \pi/2, \quad \varphi = 0, \quad \psi = \psi_0 + nt$$

$$\cos 2\psi_0 = \frac{-(1+\nu)}{2\sqrt{1-\nu+\nu^2}}, \quad \sin 2\psi_0 = \frac{(1-\nu)\sqrt{3}}{2\sqrt{1-\nu+\nu^2}} \tag{3.2}$$

(for the system Earth–Moon $\nu = 1/81.3$ and $\psi_0 = 60°18'24''$).

Solution (3.1) implies that the center of mass of body M lies at the triangular libration point L_4 of the limited circular problem of three bodies, and the axes of inertia retain a fixed position in the rotating coordinate system $Gxyz$. Axes $O\xi$, $O\zeta$ lie in the plane of the orbit, with axis $O\xi$ remaining at the constant angle ψ_0 to the moving axis Gx (Fig.1).

Let us now assume that the body is a homogeneous ellipsoid with semiaxes $a > c > b$. The force function U_0' is now, as previously, determined by formulas (2.2) and (2.3), while for force functions U_1', U_2' we have with the stated accuracy the following expressions:

$$U'_{1,2} = \frac{fm_{1,2}}{8a_0^3}\{\xi^2(-\cos^2\psi_0 + 5\sin^2\psi_0 \pm 3\sqrt{3}\sin 2\psi_0) +$$
$$\zeta^2(-\sin^2\psi_0 + 5\cos^2\psi_0 \mp 3\sqrt{3}\sin 2\psi_0) - 6\xi\zeta(\sin 2\psi_0 \pm \sqrt{3}\cos 2\psi_0) - 4\eta^2\}$$

where ψ_0 is the constant angle defined by formula (3.2). As the result the first integral of hydrodynamic equations assumes the form

$$D/\rho_0 = \Psi_1\xi^2 + \Psi_2\eta^2 + \Psi_3\zeta^2 + \Psi_4\xi + \Psi_5\zeta + \Psi_6\xi\eta + \text{const}$$

$$\Psi_1 = \frac{3}{4}n^2 - \frac{3f}{8a_0^3}[(m_1+m_2)\cos 2\psi_0 + \sqrt{3}(m_2-m_1)\sin 2\psi_0] - fF_1$$

$$\Psi_2 = -\frac{1}{2}n^2 - fF_2$$

$$\Psi_3 = \frac{3}{4}n^2 + \frac{3f}{8a_0^3}[(m_1+m_2)\cos 2\psi_0 + \sqrt{3}(-m_1+m_2)\sin 2\psi_0] - fF_3$$

$$\Psi_4 = \left[\frac{n^2a_0(1-\nu)}{2(1+\nu)} - \frac{fm_1}{2a_0^2} + \frac{fm_2}{2a_0^2}\right]\cos\psi_0 - \left[\frac{n^2a_0\sqrt{3}}{2} - \frac{\sqrt{3}f}{2a_0^2}(m_1+m_2)\right]\sin\psi_0$$

$$\Psi_5 = \left[\frac{n^2a_0(1-\nu)}{2(1+\nu)} - \frac{fm_1}{2a_0^2} + \frac{fm_2}{2a_0^2}\right]\sin\psi_0 + \left[\frac{n^2a_0\sqrt{3}}{2} - \frac{\sqrt{3}f}{2a_0^2}(m_1+m_2)\right]\cos\psi_0$$

$$\Psi_6 = -\frac{3fm_1}{4a_0^3}(\sin 2\psi_0 + \sqrt{3}\cos 2\psi_0) - \frac{3fm_2}{4a_0^3}(\sin 2\psi_0 - \sqrt{3}\cos 2\psi_0)$$

The conditions of existence of ellipsoidal equilibrium patterns of the body located at a triangular libration point is determined in conformity with solution (3.1) by the system of equalities

$$\Psi_1 a^2 = \Psi_2 b^2 = \Psi_3 c^2, \quad \Psi_4 = \Psi_5 = \Psi_6 = 0$$

It can be proved that in the case of the obtained value of ψ_0 (3.2) $\Psi_6 = 0$, while the quantities Ψ_4, Ψ_5 vanish by virtue of the solution of equations of the translational-rotational motion (3.1). The remaining equalities reduce to the form

$$({}^3/_4 n^2[1+\delta(\nu)] - fF_1)a^2 = (-{}^1/_2 n^2 - fF_2)b^2 =$$
$$({}^3/_4 n^2[1-\delta(\nu)] - fF_3)c^2, \quad \delta(\nu) = \frac{\sqrt{1-\nu+\nu^2}}{1+\nu} \tag{3.3}$$

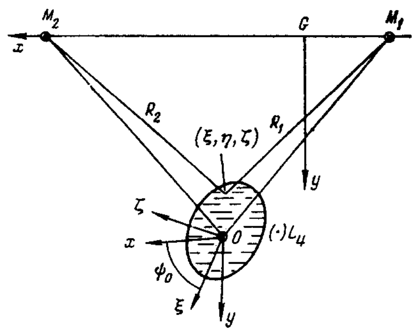

Fig.1

which for $\nu = 0$ coincide with the known conditions of existence of Roche's ellipsoids /5/ and determine a sequence of ellipsoidal equilibrium patterns for the fluid body located at a triangular libration point L_4.

Remarks. 1°. Condition $\mu = 0$ implies that terms of order $(\bar{R}/\bar{r})^2$ (a very small quantity in the case of real celestial bodies) are neglected in the right-hand sides of Eqs.(1.1).

2°. When $\mu = 0$, the rotational motion of axes of inertia of body M are determined by the second harmonics of force functions of bodies M, M_1 and M, M_2 (in this case the force function U is defined by formula (1.4) when $W = 0$).

3°. When $\mu = 0$ for a given circular motion of relative equilibrium of the fluid body, the problem of its ellipsoidal equilibrium patterns is formulated as the Roche's problem /5/.

4°. For the derivation of exact particular solutions of equations of motion, and investigation of their stability the methods in /2,5/ and others may be used.

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